

Grassmannian flows and applications to nonlinear partial differential equations

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16th February 2017

Abstract We show how solutions to a large class of Riccati evolutionary nonlinear partial differential equations can be generated from the corresponding linearized equations. The key is an integral equation analogous to the Marchenko equation, or more generally dressing transformation, in integrable systems. We show explicitly how this can be achieved for scalar partial differential equations with nonlocal quadratic nonlinearities. We provide numerical simulations that demonstrate the generation of solutions to a Fisher–Kolmogorov–Petrovskii–Piskunov type equation with a nonlocal nonlinearity from arbitrary initial data using this approach. We also indicate how the method might extend to more general classes of nonlinear partial differential systems.

1 Introduction

It is well-known that solutions to many integrable nonlinear partial differential equations can be generated from solutions to a linear integrable equation namely the Gel’fand–Levitan–Marchenko equation. It is an example of a generic dressing transformation which we shall express in the form

$$g(x, y) = p(x, y) + \int_x^\infty g(x, z)q(z, y; x) dz,$$

for $y \geq x$. See Zakharov and Shabat [30] or Dodd, Eilbeck, Gibbon and Morris [8] for more details. Here all the functions shown may depend explicitly on time t , and we suppose that q and p represent given data and g is the solution. Typically p represents the scattering data and takes the form $p = p(x + y)$ while q depends on p , for example $q = p$ in the case of the Korteweg de Vries equation. See Ablowitz, Ramani and Segur [2] for

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more details. Typically given a nonlinear integrable partial differential equation, the function p is the solution to an associated linear system and the solution to the nonlinear integrable equation is given by $u = -2(d/dx)g(x, x)$. See for example Drazin and Johnson [9, p. 86] for the case of the Korteweg de Vries equation. The notion that the solution to a corresponding linear partial differential equation can be used to generate solutions to nonlinear integrable partial differential equations is addressed in the review by Miura [17]. An explicit formula was provided by Dyson [10] who showed that for the Korteweg de Vries equation the solution to the Gel'fand–Levitan–Marchenko equation along the diagonal $g = g(x, x)$ can be expressed in terms of the derivative of the logarithm of a tau-function or Fredholm determinant. In a series of papers Pöppe [19, 20, 21], Pöppe and Sattinger [22], Bauhardt and Pöppe [4], and Tracy and Widom [28] expressed the solutions to further nonlinear integrable partial differential equations in terms of Fredholm determinants. Importantly Pöppe [19] explicitly states the idea that:

“For every soliton equation, there exists a *linear* PDE (called a base equation) such that a map can be defined mapping a solution p of the base equation to a solution u of the soliton equation. The properties of the soliton equation may be deduced from the corresponding properties of the base equation which in turn are quite simple due to linearity. The map $p \rightarrow u$ essentially consists of constructing a set of linear integral operators using p and computing their Fredholm determinants.”

From our perspective, the solution g to the dressing transformation represents an element of a Fredholm Grassmann manifold, expressed in a given coordinate patch. The basic idea is as follows; we proceed formally. Let $Q = \text{id}$, P and G be linear operators with corresponding integral kernels $q = q(x, y; t)$, $p = p(x, y; t)$ and $g = g(x, y; t)$, which are in general time dependent. We re-write the dressing transformation above in the *general* form

$$P = GQ.$$

Note we have changed the sign of q and in general no longer necessarily suppose the integral operation implicit in GQ is of Volterra type. We assume Q is a Fredholm operator, i.e. a compact perturbation of the identity, while P and G are compact operators. Now suppose the operators Q and P satisfy the following linear operator system for $t \geq 0$,

$$\begin{aligned}\partial_t Q &= AQ + BP, \\ \partial_t P &= CQ + DP.\end{aligned}$$

Here we assume, typically, that A and C are bounded linear operators while B and D are unbounded linear operators. Differentiating the relation $P = GQ$ with respect to time using the product rule and equivalencing by Q , i.e. post-composing by Q^{-1} , direct formal calculation implies

$$\partial_t G = C + DG - G(A + BG).$$

This is a Riccati evolution equation for the operator G . We now come to the *crucial* component steps of this paper. The Riccati evolution equation for G in principle corresponds to an evolution equation for g . In particular we can pick out the evolution equation satisfied by, for example, the diagonal component of g . We express the resulting equation governing the evolution of the diagonal component of $g = g(x, x; t)$ in the form

$$\partial_t g = c + dg - g \star (a + bg).$$

Here c and a are now the integral kernels corresponding to the operators A and C above, while $b = B$ and $d = D$. The character of the operator G determines the form of this eventual evolution equation for $g = g(x, x; t)$, and the natural question is: what form do the terms like $d g$ take and what is the character of the product operation \star shown. Before we answer this let us summarize the procedure we are proposing. We use the following nomenclature for the linear partial differential equations prescribing the evolution of the kernels q and p corresponding to the operators $Q - \text{id}$ and P respectively, as well as the map G with kernel g between them:

1. *Base equation*: $\partial_t p = c \star (\delta + q) + d p$;
2. *Auxiliary equation*: $\partial_t q = a \star (\delta + q) + b p$;
3. *Riccati mapping*: $p = g \star (\delta + q)$.

Here the δ is the delta function representing the identity at the level of integral operators. The reason underlying the nomination of the base and auxiliary equations is that in most of our examples we take $a \equiv c \equiv 0$ —let us assume this with loss of generality for our present argument. Then our goal is as follows. Given a quadratically nonlinear partial differential equation, can we re-engineer solutions to it from solutions to the corresponding base and auxiliary equations. The solutions are re-engineered/generated by solving the Riccati mapping which is a linear Fredholm integral equation. For this procedure to be successful we need to identify the operators d and b and determine the appropriate character for the operator G so that the partial differential equation $\partial_t g = d g - g \star (b g)$ corresponds to the original quadratically nonlinear partial differential equation.

As is well-known, this procedure works for most integrable systems, as demonstrated in Ablowitz, Ramani and Segur [2] who assume $p = p(x + y)$ is a Hankel kernel. For example, we can generate solutions to the Korteweg de Vries equation from the Gel'fand–Levitan–Marchenko equation by setting $q \equiv p$. Note that in our setting if $b = b(\partial)$ and $d = d(\partial)$ are monomials in $\partial = \partial_x$ then the base and auxiliary equations imply that $q = b(\partial) d(\partial^{-1}) p$, i.e. a direct relation between q and p . As another example, we can generate solutions to the nonlinear Schrödinger equation by assuming $q(z, y; x) = \pm \int_x^\infty \bar{p}(z, \zeta) p(\zeta, y) d\zeta$ where \bar{p} represents the complex conjugate of p . In this case it is also well known that such solutions can be generated from a 2×2 matrix-valued dressing transformation.

In this paper the main example we consider is a Fisher–Kolmogorov–Petrovskii–Piskunov type equation with a nonlocal nonlinearity of the form $g \star g$ where the operation \star denotes convolution. Indeed we show how solutions can be generated using the approach we propose from arbitrary initial data. Indeed from this example we can see straightforwardly how our procedure extends to any higher order diffusion, for example fourth order or higher, or indeed any operator $d = d(\partial)$ polynomial in ∂ with constant coefficients. We provide numerical simulations to confirm this. We also consider another important example.

Our paper is structured as follows. In Section 2 we show how linear subspace flows induce Riccati flows in coordinate patches of the corresponding Fredholm Grassmannian. We derive the equation for the evolution of the integral kernel associated with the Riccati flow. We then consider two pertinent examples in Section 3 of quadratically nonlinear partial differential equations, i.e. Riccati-type partial differential equations. Their solutions can be derived by solving the linear base and auxiliary partial differential equations (the subspace flow) and then solving the linear Fredholm equation representing the projection of the subspace flow onto a coordinate patch of the Fred-

holm Grassmannian. Then finally in Section 4 we discuss possible extensions of our solution approach to other classical quadratically nonlinear partial differential equations.

2 Fredholm Grassmannian flows

Suppose we have a separable Hilbert space \mathbb{H} that admits a direct sum decomposition $\mathbb{H} = \mathbb{Q} \oplus \mathbb{P}$, where \mathbb{Q} and \mathbb{P} are closed subspaces of \mathbb{H} . The Fredholm Grassmannian $\text{Gr}(\mathbb{H}, \mathbb{Q})$ is the set of all subspaces that are ‘comparable in size’ with \mathbb{Q} . More precisely, the Fredholm Grassmannian $\text{Gr}(\mathbb{H}, \mathbb{Q})$ is the set of all closed subspaces \mathbb{W} of \mathbb{H} such that: (i) the orthogonal projection $\mathbb{W} \rightarrow \mathbb{Q}$ is a Fredholm operator, and (ii) the orthogonal projection $\mathbb{W} \rightarrow \mathbb{P}$ is a Hilbert–Schmidt operator. Coordinate patches of $\text{Gr}(\mathbb{H}, \mathbb{Q})$ are graphs of operators $\mathbb{Q} \rightarrow \mathbb{P}$ parameterized by, say, G . See Sato [26] and Pressley and Segal [23] for more details.

We begin by defining two linear operators $Q: \mathbb{Q} \rightarrow \mathbb{Q}$ and $P: \mathbb{Q} \rightarrow \mathbb{P}$. We assume that Q is a Fredholm operator such that $Q - \text{id}$ is of Hilbert–Schmidt class. We also assume P is a compact operator of Hilbert–Schmidt class. We suppose \mathbb{H} is a Sobolev function space on \mathbb{R} , and $Q - \text{id}$ and P have integral kernels q and p as follows. For any function $f \in \mathbb{Q}$ we set

$$(Q - \text{id})(f)(x; t) := \int_{\mathbb{I}} q(x, y; t) f(y) dy,$$

$$P(f)(x; t) := \int_{\mathbb{I}} p(x, y; t) f(y) dy,$$

where $x \in \mathbb{R}$. These operators and their kernels depend on a time parameter $t \geq 0$. We assume the interval \mathbb{I} is the real line or $[a, \infty)$ for some real constant a .

We now specify the evolution of the operators Q and P .

Definition 1 (Auxiliary and Base Equations) We suppose the operators Q and P satisfy the linear system of operator equations

$$\begin{aligned} \partial_t Q &= A Q + B P, \\ \partial_t P &= C Q + D P, \end{aligned}$$

or equivalently

$$\begin{aligned} \partial_t q &= a \star (\delta + q) + b p, \\ \partial_t p &= c \star (\delta + q) + d p, \end{aligned}$$

for $q = q(x, y; t)$ and $p = p(x, y; t)$. We call the evolution equation for q the *auxiliary equation* and that for p the *base equation*. Here we assume $b \equiv B$ and $d \equiv D$ are linear operators $b = b(\partial_x)$ and $d = d(\partial_x)$ that are polynomial functions of ∂_x . We assume A and C to be linear operators with integral kernels a and c , respectively, while

$$(a \star (\delta + q))(x, y) := \int_{\mathbb{I}} a(x, z) (\delta(z - y) + q(z, y; t)) dz,$$

or equivalently

$$(a \star (\delta + q))(x, y) := a(x, y; t) + \int_{\mathbb{I}} a(x, z) q(z, y; t) dz.$$

and similarly for c .

The Auxiliary and Base Equations represent two of the essential ingredients in our prescription, which to be complete, requires a third crucial ingredient. This is to propose a relation between Q and P as follows.

Definition 2 (Riccati Mapping Relation) Let $G: \mathbb{Q} \rightarrow \mathbb{P}$ be a Hilbert–Schmidt operator with kernel g of the form

$$G(f)(x; t) := \int_{\mathbb{I}} g(x, y; t) f(y) dy.$$

We define the Riccati Mapping Relation as the operator equation given by $P = GQ$ relating Q and P which has the following precise form

$$p(x, y; t) = g(x, y; t) + \int_{\mathbb{I}} g(x, z; t) q(z, y; t) dz.$$

We are now in a position to establish our main result.

Theorem 1 (Grassmannian evolution equation) *Suppose linear operators Q and P satisfy the linear system of auxiliary and base operator equations above and are related by the linear operator G described above so that $P = GQ$. Then the integral kernel function $g = g(x, y; t)$, associated with the linear operator G , necessarily satisfies the equation*

$$\partial_t g(x, y; t) = c(x, y; t) + d(\partial_x) g(x, y; t) - \int_{\mathbb{I}} g(x, z; t) (a(z, y; t) + b(\partial_z) g(z, y; t)) dz.$$

Proof First we differentiate the proposed relation $P = GQ$ with respect to time. This generates the relation

$$(\partial_t G) Q = \partial_t P - G(\partial_t Q).$$

Using the kernel form for the Riccati mapping relation above this has the form

$$\partial_t g(x, y; t) + \int_{\mathbb{I}} \partial_t g(x, z; t) q(z, y; t) dz = \partial_t p(x, y; t) - \int_{\mathbb{I}} g(x, z; t) \partial_t q(z, y; t) dz.$$

Second we substitute for $\partial_t q$ and $\partial_t p$ using their evolution equations. Let us consider the first term on the right above. We find that

$$\begin{aligned} \partial_t p(x, y; t) &= \int_{\mathbb{I}} c(x, z; t) (\delta(z - y) + q(z, y; t)) dz + d(\partial_x) p(x, y; t) \\ &= \int_{\mathbb{I}} c(x, z; t) (\delta(z - y) + q(z, y; t)) dz \\ &\quad + d(\partial_x) \left(g(x, y; t) + \int_{\mathbb{I}} g(x, z; t) q(z, y; t) dz \right) \\ &= \int_{\mathbb{I}} c(x, z; t) (\delta(z - y) + q(z, y; t)) dz \\ &\quad + d(\partial_x) \int_{\mathbb{I}} g(x, z; t) (\delta(z - y) + q(z, y; t)) dz \\ &= \int_{\mathbb{I}} (c(x, z; t) + d(\partial_x) g(x, z; t)) (\delta(z - y) + q(z, y; t)) dz. \end{aligned}$$

Now consider the second term on the right above. We observe

$$\begin{aligned}
& \int_{\mathbb{I}} g(x, z; t) \partial_t q(z, y; t) \, dz \\
&= \int_{\mathbb{I}} g(x, z; t) \left(\int_{\mathbb{I}} a(z, \zeta; t) (\delta(\zeta - y) + q(\zeta, y; t)) \, d\zeta \right) dz \\
&\quad + \int_{\mathbb{I}} g(x, z; t) (b(\partial_z) p(z, y; t)) \, dz \\
&= \int_{\mathbb{I}} g(x, z; t) \left(\int_{\mathbb{I}} a(z, \zeta; t) (\delta(\zeta - y) + q(\zeta, y; t)) \, d\zeta \right) dz \\
&\quad + \int_{\mathbb{I}} g(x, z; t) \left(b(\partial_z) \int_{\mathbb{I}} g(z, \zeta; t) (\delta(\zeta - y) + q(\zeta, y; t)) \, d\zeta \right) dz \\
&= \int_{\mathbb{I}} \left(\int_{\mathbb{I}} g(x, \zeta; t) (a(\zeta, z; t) + b(\partial_\zeta) g(\zeta, z; t)) \, d\zeta \right) (\delta(z - y) + q(z, y; t)) \, dz.
\end{aligned}$$

Putting these results together and equivalencing by the action of Q , i.e. post-composing by Q^{-1} , generates the required result. \square

Remark 1 We make the following important observations:

1. We have shown that if the operators P and Q , parameterized by p and q respectively, satisfy a linear system of operator equations, then the operator $G: Q \rightarrow P$, parameterized by g , satisfies the Riccati operator differential equation shown.
2. The proof reproduces at the integral kernel level, the direct procedure of differentiating the operator relation $P = GQ$ with respect to time, and substituting for the time derivatives of P and Q from the base and auxiliary equations.
3. To generate a quadratically nonlinear partial differential equation from the equation for $g = g(x, y; t)$ in the theorem above, we need to consider a trace or functional of G , for example we could set $y = x$ and examine the partial differential equation governing the evolution of $g = g(x, x; t)$.
4. We need to demonstrate as a practical procedure, how linear partial differential equations for p and q generate solutions to the quadratically nonlinear partial differential equation at hand. See the next section for the details of this procedure in some important cases.
5. The character of the operator G plays a crucial role in 2 and 3 just above.

3 Examples

We now show how some quadratically nonlinear partial differential equations can be generated via the approach outlined in the last section. For each example we show how solutions to the quadratically nonlinear equations concerned can in practice be generated from linear base and auxiliary equations. Note that throughout we define the Fourier transform for any given function $f = f(x)$ and its inverse as

$$\hat{f}(k) := \int_{\mathbb{R}} f(x) e^{2\pi i k x} \, dx \quad \text{and} \quad f(x) := \int_{\mathbb{R}} \hat{f}(k) e^{-2\pi i k x} \, dk.$$

Example 1 (Nonlocal quadratic nonlinearity) We begin by assuming that $\mathbb{I} := \mathbb{R}$ and the kernel g of the operator G has the convolution form $g = g(x - y; t)$. We assume the linear base and auxiliary equations have the form

$$\begin{aligned}\partial_t p(x, y; t) &= d p(x, y; t), \\ \partial_t q(x, y; t) &= b p(x, y; t),\end{aligned}$$

We assume $b \equiv 1$, i.e. simple multiplication by unity, while in general $d = d(\partial_x)$. In this case the Grassmannian evolution equation in Theorem 1 has the form

$$\partial_t g(x - y; t) = d(\partial_x) g(x - y; t) - \int_{\mathbb{R}} g(x - z; t) g(z - y; t) dz.$$

By setting $y = 0$ we arrive at the nonlocal quadratically nonlinear partial differential equation

$$\partial_t g = d(\partial) g - g \star g,$$

where the \star operation here does indeed represent convolution. In Fourier space this naturally takes the form

$$\partial_t \hat{g} = d(2\pi i k) \hat{g} - \hat{g}^2.$$

We can generate solutions to the partial differential equation for g from the linear base and auxiliary equations, for any given initial data, as follows. Note the base equation has the following equivalent form and solution in Fourier space:

$$\partial_t \hat{p} = d(2\pi i k) \hat{p} \quad \Leftrightarrow \quad \hat{p}(k, y; t) = e^{d(2\pi i k) t} \hat{p}_0(k, y).$$

Here \hat{p}_0 is the Fourier transform of the initial data for p ; we show how to generate this presently. In Fourier space the auxiliary equation has the form and solution:

$$\partial_t \hat{q} = \hat{p} \quad \Leftrightarrow \quad \hat{q}(k, y; t) - \hat{q}_0(k, y) = \frac{e^{d(2\pi i k) t} - 1}{d(2\pi i k)} \hat{p}_0(k, y).$$

Here $\hat{q}_0(k, y)$ is the Fourier transform of the initial data for \hat{q} , which we take to be *identically zero*. This means if we set $t = 0$ in the Riccati Mapping Relation we find

$$p_0(x, y) = g_0(x - y) \quad \Leftrightarrow \quad \hat{p}_0(k, y) = e^{2\pi i k y} \hat{g}_0(k).$$

where g_0 is the initial data for the partial differential equation for g . Hence explicitly we have

$$\hat{p}(k, y; t) = e^{d(2\pi i k) t} e^{2\pi i k y} \hat{g}_0(k) \quad \text{and} \quad \hat{q}(k, y; t) = \frac{e^{d(2\pi i k) t} - 1}{d(2\pi i k)} e^{2\pi i k y} \hat{g}_0(k).$$

Note by taking the inverse Fourier transform, we deduce that $p = p(x - y; t)$ and $q = q(x - y; t)$. Further note the Riccati relation in this case is

$$\begin{aligned}p(x, y; t) &= g(x - y; t) + \int_{\mathbb{R}} g(x - z; t) q(z, y; t) dz \\ \Leftrightarrow \quad \hat{p}(k, y; t) &= \hat{g}(k; t) (e^{2\pi i k y} + \hat{q}(k, y; t)).\end{aligned}$$

Thus using the expressions for \hat{p} and \hat{q} above we find that

$$\hat{g}(k; t) = \frac{e^{d(2\pi i k) t} \hat{g}_0(k)}{1 + \left((e^{d(2\pi i k) t} - 1) / d(2\pi i k) \right) \hat{g}_0(k)}.$$

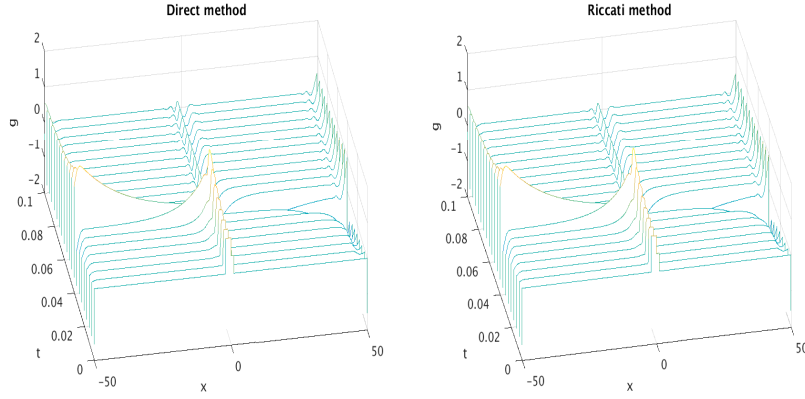


Fig. 1 We plot the solution to the nonlocal quadratically nonlinear partial differential equation from Example 1. We used a generic initial profile g_0 as shown. The left panel shows the solution computed using a direct integration approach while the right panel shows the solution computed using our Riccati approach.

In Figure 1 we show the solution to the nonlocal quadratically nonlinear partial differential equation above, for $d = \partial_x^2 + 1$ and a given generic initial profile g_0 . The left panel in the figure shows the evolution of the solution profile computed using a direct integration approach. By this we mean we approximated ∂_x^2 by the central difference formula and computed the nonlinear convolution by computing the inverse Fourier transform of $(\hat{g}(k))^2$. We used the inbuilt Matlab integrator `ode23s` to integrate in time. Similar direct integration could be achieved by integrating the differential system for \hat{g} using `ode23s` and then computing the inverse Fourier transform. The right panel in Figure 1 shows the solution evolution computed using the Riccati mapping procedure. As expected, the solutions are identical, even when we continue the solution past the time when the diffusion has reached the finite computation boundaries roughly half way along the interval of evolution shown.

Remark 2 This last example extends to the case where $x, y \in \mathbb{R}^n$ for any $n \geq 1$ when d is a scalar operator such as a power of the Laplacian, with p, q and g all scalar.

Example 2 (Nonlocal quadratic nonlinearity with correlation) For this case we assume $\mathbb{I} := \mathbb{R}$ and the kernel g of the operator G has the form $g = g(x + \xi, y + \xi; t)$ for some parameter $\xi \in \mathbb{R}$. We assume the linear base and auxiliary equations have the form

$$\begin{aligned}\partial_t p(x, y; \xi, t) &= d(\partial_1) p(x, y; \xi, t), \\ \partial_t q(x, y; \xi, t) &= b(x) p(x, y; \xi, t).\end{aligned}$$

In particular we will assume $b = N(x, \sigma)$ where $N = N(x, \sigma)$ is the Gaussian probability density function with mean zero and standard deviation σ . The operator ∂_1 refers to the partial derivative with respect to the first argument. In this case the Riccati relation has the form

$$p(x, y; \xi, t) = g(x + \xi, y + \xi; t) + \int_{\mathbb{R}} g(x + \xi, z + \xi; t) q(z, y; \xi, t) dz.$$

By slightly modifying the proof of the Grassmannian evolution equation in Theorem 1, the corresponding nonlocal partial differential equation generated for $g = g(x + \xi, y + \xi; t)$ is

$$\begin{aligned} \partial_t g(x + \xi, y + \xi; t) \\ = d(\partial_1) g(x + \xi, y + \xi; t) - \int_{\mathbb{R}} g(x + \xi, z + \xi; t) b(z) g(z + \xi, y + \xi; t) dz. \end{aligned}$$

By setting $X := x + \xi$, $Y := y + \xi$ and $Z := z + \xi$ we can also express this in the form

$$\partial_t g(X, Y; t) = d(\partial_X) g(X, Y; t) - \int_{\mathbb{R}} g(X, Z; t) b(Z - \xi) g(Z, Y; t) dZ.$$

Here we have suppressed the fact that g will also depend on ξ as in our actual application we will set $y = 0$ or equivalently $\xi = Y$.

We can generate solutions to the equation for $g = g(x + \xi, y + \xi; t)$ or the equivalent form $g = g(X, Y; t)$ from the linear base and auxiliary equations, for any given initial data, as follows. In Fourier space the solution of the base equation has the form

$$\hat{p}(k, y; \xi, t) = e^{d(2\pi i k) t} \hat{p}_0(k, y; \xi),$$

where \hat{p}_0 is the Fourier transform of the initial data for p . The auxiliary equation solution in Fourier space has the form,

$$\hat{q}(k, y; \xi, t) = \int_{\mathbb{R}} \hat{b}(k - \kappa) \hat{I}(\kappa, t) \hat{p}_0(\kappa, y; \xi) d\kappa,$$

where we set

$$\hat{I}(k, t) := \frac{e^{d(2\pi i k) t} - 1}{d(2\pi i k)}.$$

As in the last example we take the initial data for q to be zero and thus the initial data for \hat{q} is also zero. We now set $t = 0$ in the Riccati Mapping Relation and find that in Fourier space

$$\hat{p}_0(k, y; \xi) = e^{-2\pi i k \xi} \hat{g}_0(k, y + \xi).$$

where g_0 is the initial data for the partial differential equation for g . Hence explicitly we have

$$\hat{p}(k, y; \xi, t) = e^{d(2\pi i k) t} e^{-2\pi i k \xi} \hat{g}_0(k, y + \xi)$$

and

$$\hat{q}(k, y; \xi, t) = \int_{\mathbb{R}} \hat{b}(k - \kappa) \hat{I}(\kappa, t) e^{-2\pi i \kappa \xi} \hat{g}_0(\kappa, y + \xi) d\kappa.$$

Note that we can deduce at this stage that $p = p(x + \xi, y + \xi; t)$.

We now derive an explicit form for q from $\hat{q} = \hat{q}(k, y; \xi, t)$ above. Taking the inverse Fourier transform of \hat{q} and using that, as the Gaussian probability density function with mean zero, $b = N(x, \sigma)$ and its Fourier transform are symmetric, we find that

$$\begin{aligned} q(x, y; \xi, t) &= \int_{\mathbb{R}} e^{-2\pi i \kappa \xi} \left(\int_{\mathbb{R}} e^{-2\pi i k x} \hat{b}(k - \kappa) dk \right) \hat{I}(\kappa, t) \hat{g}_0(\kappa, y + \xi) d\kappa \\ &= \int_{\mathbb{R}} e^{-2\pi i \kappa \xi} (e^{-2\pi i \kappa x} b(x)) \hat{I}(\kappa, t) \hat{g}_0(\kappa, y + \xi) d\kappa \\ &= b(x) \int_{\mathbb{R}} e^{-2\pi i \kappa (\xi + x)} \hat{I}(\kappa, t) \hat{g}_0(\kappa, y + \xi) d\kappa \\ &= b(x) \int_{\mathbb{R}} I(x + \xi - \zeta, t) g_0(\zeta, y + \xi) d\zeta. \end{aligned}$$

Let us now consider the Riccati relation. Taking the Fourier transform of the Riccati relation, substituting in the expression for \hat{p} for the moment and dividing through by $e^{-2\pi i k \xi}$, we find

$$e^{d(2\pi i k) t} \hat{g}_0(k, y + \xi) = \hat{g}(k, y + \xi; t) + \int_{\mathbb{R}} \hat{g}(k, z + \xi; t) q(z, y; \xi, t) dz.$$

This is equivalent to the form, and trivial observation, that we could have written,

$$p(x, y + \xi, t) = g(x, y + \xi; t) + \int_{\mathbb{R}} g(x, z + \xi; t) q(z, y; \xi, t) dz.$$

Inserting the expression for q above into this for of the Riccati relation we find

$$\begin{aligned} p(x, y + \xi, t) &= g(x, y + \xi; t) + \int_{\mathbb{R}} g(x, z + \xi; t) b(z) \int_{\mathbb{R}} I(z + \xi - \zeta, t) g_0(\zeta, y + \xi) d\zeta dz \\ &= g(x, y + \xi; t) + \int_{\mathbb{R}} g(x, Z; t) b(Z - \xi) \int_{\mathbb{R}} I(Z - \zeta, t) g_0(\zeta, y + \xi) d\zeta dZ, \end{aligned}$$

where we used the substitution $Z := z + \xi$. Now set $y = 0$ so $\xi = Y$, we find that

$$p(x, Y; t) = g(x, Y; t) + \int_{\mathbb{R}} g(x, Z; t) B(Z, Y) dZ.$$

where

$$B(Z, Y) := b(Z - Y) \int_{\mathbb{R}} I(Z - \zeta, t) g_0(\zeta, Y) d\zeta.$$

We can recover an explicit expression for $p = p(x, Y; t)$ from its Fourier transform above. Hence we have a Fredholm equation for $g = g(x, Y; t)$ which we can in principle solve. Indeed we solved the Fredholm equation numerically. The results are shown in Figure 2. We set the operator $d = \partial_x^2 + 1$ and took as the generic initial profile $g_0(x, y) := \text{sech}(x + y) \text{sech}(y)$. The top panel in Figure 2 shows the initial data. The middle panel in the figure shows the solution profile computed at time $t = 2$ using a direct spectral integration approach. By this we mean we solved the equation for $\hat{g}(k, Y; t)$ generated by taking the Fourier transform of the equation for $g = g(x, Y; t)$. We used the inbuilt Matlab integrator `ode23s` to integrate in time. The bottom panel in Figure 2 shows the solution computed with the time parameter $t = 2$ using the Riccati mapping procedure, i.e. by numerically solving the Fredholm equation for $g = g(x, Y; t)$ above by standard methods for such integral equations. As expected, the solutions in the middle and bottom panels are identical.

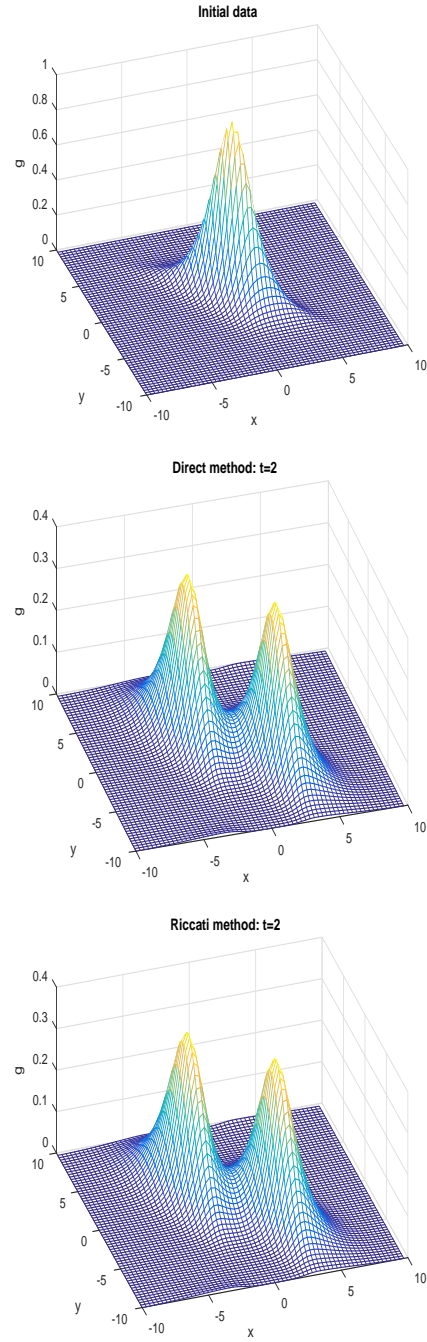


Fig. 2 We plot the solution to the nonlocal quadratically nonlinear partial differential equation with correlation from Example 2. We used a generic initial profile g_0 as shown in the top panel. For time $t = 2$, the middle panel shows the solution computed using a direct integration approach while the bottom panel shows the solution computed using our Riccati approach.

Remark 3 We make the following observations:

1. It is important to emphasize that in the Riccati approach the role of time t is as a parameter. One decides the time at which one wants to compute the solution and we solve the Fredholm equation for that single time t .
2. In both examples we take the initial data for q to be zero. This is motivated as follows. We think of the restricted general linear group acting on the Fredholm Grassmannian much like $GL(n)$ acts on the Grassmannian of say k -planes in n -space. The subgroup of transformations from $\mathbb{Q} \rightarrow \mathbb{Q}$ parameterized by Q corresponds to the subgroup $GL(k)$ of $GL(n)$, and represents a component of its action. Thus considering the Grassmannian as a homogeneous manifold we can take $Q = \text{id}$ at time $t = 0$ which corresponds to $q = 0$.
3. If $Q - \text{id}$ is compact on \mathbb{Q} , then by the Fredholm alternative, either Q^{-1} exists or $GQ = O$ has a solution, where O is the null operator from \mathbb{Q} to \mathbb{P} . The latter implies P is trivial, which we disregard. Hence we can solve for $G = PQ^{-1}$ and thus g . See for example Reed and Simon [24, p. 203] for more details.
4. Burgers equation is a very well-known and special case. Cole–Hopf solutions in \mathbb{R} can be found as a ratio of functions whose denominator satisfies the heat equation and the numerator is ‘-2’ times the derivative of the denominator. This suggests there is a rank one relation between p and q . There are multiple ways to express such a rank one relation, for example by assuming the Riccati relation is of Volterra type. Presently it suffices to say that indeed in \mathbb{R}^d irrotational solutions of the form $g(x; t) = -2p(x; t)/q(x; t)$, where p is the gradient of q , are naturally generated.

4 Conclusions

There are many extensions and directions of investigation we intend to follow. One interesting path is the following case. Consider the nonlocal partial differential equation for $g = g(x + \xi, y + \xi; t)$ given in Example 2. Recall we set $b = N(x, \sigma)$, the mean-zero normal probability density function. If we set $x = y = 0$, $d = \partial_1^2 + 1$ and considered the limit $\sigma \rightarrow 0$ so $b \rightarrow \delta(x)$, we would recover the original Fisher–Komogorov–Petrovskii–Piskunov equation with quadratic pointwise nonlinearity and diffusion coefficient $1/4$. However as we saw in Example 2 the function q , once explicitly calculated, has $b = N(x, \sigma)$ as a factor. This means, once this is substituted into the Riccati relation, that the operator G collapses to a rank 1 operator. In other the words the Riccati approach boils down to searching for rational function solutions to the FKPP equation, which is a too restrictive class. However we can in principle use the finite σ to try to approximate solutions to the FKPP equation. Another approach here is to suppose $g = g(x + y + 2\xi)$ is Hankel operator, a special case of that just above, but with $\mathbb{I} := [0, \infty)$ and $b = -2\partial_x$. This again generates the FKPP equation, but unfortunately, with q initially zero, this again reduces the operator G to a rank one operator and a search for simple rational solutions.

There are two immediate extensions we intend to consider next. These are the extension of the procedure above to systems of nonlinear partial differential equations and the extension to multi-dimensions. These are in addition to the full range of possible choices for integration domain in the Riccati relation, and the operators d and b both as unbounded and bounded operators. In the case of the extension to systems, as indicated in the introduction, we intend to look for the extension of our approach to higher degree nonlinearities. Lastly we remark that for the classes of nonlinear

partial differential equations we can consider, solution singularities correspond to poor choices of coordinate patches which are related to function space regularity. In principle solutions can be continued by changing coordinate patches. This is achieved by pulling back to the flow to the relevant general linear group and then projecting down to a more appropriate coordinate patch of the Fredholm Grassmannian. Alternatively, we could continue the flow in the appropriate general linear group via the base and auxiliary equations, and then monitoring the relevant projection(s).

Acknowledgements We would like to thank Percy Deift, Kurusch Ebrahimi-Fard and Anke Wiese for their extremely helpful comments and suggestions. The work of M.B. was partially supported by US National Science Foundation grant DMS-1411460.

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